Math 299 **Direct proofs involving inequalities**

Consider the following basic facts regarding inequalities.

A1 For all real numbers a, b, c, if $a \leq b$ and $b \leq c$ then $a \leq c$.

A2 For all real numbers a, b, c, if $a \leq b$ then $a + c \leq b + c$.

A3 For all real numbers a, b, c, if $a \leq b$ and $0 \leq c$ then $ac \leq bc$.

Prove the statements below using A1-A3, together with any basic facts about *equality* $=$.

1. For all real numbers a, b, if $0 \leq a$ and $a \leq b$ then $a^2 \leq b^2$.
2. For all real numbers a, if $a \leq 0$ then $0 \leq -a$.
3. For all real numbers a, b, if $b \leq a$ and $a \leq 0$, then $a^2 \leq b^2$.
4. For all real numbers b, $0 \leq b^2$.
5. For all real numbers a, b, $ab \leq \frac{1}{2}(a^2 + b^2)$. *Hint: Consider* $(a - b)^2$.
6. For all real numbers a, b, δ, if $\delta \neq 0$ then $ab \leq \frac{1}{2}(\delta^2 a^2 + \delta^{-2} b^2)$.
7. For all real numbers a, b, $ab = \frac{1}{2}(a^2 + b^2)$ if and only if $a = b$.
8. For all non-negative real numbers a, b, $\sqrt{ab} \leq \frac{1}{2}(a + b)$. This is called the *arithmetic-geometric mean inequality.*

Hints:

1. Use A3 and A1.
2. Use A2 with $c = -a$.
3. This is similar to Problem 1, except you should use Problem 2 so that A3 applies.
4. Prove using cases: $0 \leq a$ (use Problem 1) and $a \leq 0$ (use Problem 3).
5. Use the inequality $0 \leq (a - b)^2$ (why is this true?) and then FOIL.
6. Notice that $ab = (\delta a)(\delta^{-1} b)$.